Plan:

(1) Photon propagating in a semiconductor

(2) ~~General~~ Modified photon/phonon propagator

(3) generic Feynman Rules

(4) Resummation ~~of~~ → photon self energy

(1) photon propagating in a semi-conductor

Hamiltonian

$$H = H_0 + V$$

$$H_0 = \sum_k \omega_k c_k^\dagger c_k + \sum_k \tilde{\omega}_k d_k^\dagger d_k + \sum_k c|k| a_k^\dagger a_k$$

$$\cancel{V = \int dk \Big[\underset{①}{-j(k)\cdot A(-k)} + \underset{②}{\frac{e^2 n_0}{2m} A(k)\cdot A(-k)} \Big]}$$

$$\cancel{p^2 \to (p-eA)^2 = p^2 - \underset{①}{2ep\cdot A} + \underset{②}{e^2A^2}}$$

~~Defining~~

~~How to define the current?~~

What about interaction part — let's do a better job this time. From QM class, we remember that

$$H = [\text{Potential energy Part}] + \underbrace{\int dr \Big[\cancel{\tfrac{p^2}{2m}}\ \psi^\dagger_{(r)} \frac{p^2}{2m} \psi(r) \Big)}_{\text{Kinetic energy}}$$

let us focus on KE part:

$$[KE] = \int dr\, \psi^\dagger \frac{p^2}{2m} \psi \rightarrow \boxed{\text{minimal substitution } p \rightarrow p - eA}$$

$$[\tilde{KE}] = \int dr \frac{1}{2m}\psi^\dagger (p-eA)^2 \psi = \int dr \left[\psi^\dagger \frac{p^2}{2m}\psi + \psi^\dagger\left(-\frac{peA}{2m} - \frac{eAp}{2m} + \frac{e^2A^2}{2m}\right)\psi\right]$$

$$[\tilde{KE}] = [KE] + \int dr\, \psi^\dagger\left(-\frac{e\nabla}{2mi}\cdot A - A\cdot\frac{e\nabla}{2mi} + \frac{e^2A^2}{2m}\right)\psi$$

move to other side to get V_{int}

this term contains $\vec{\nabla}\cdot A\psi$, we would like ∇ to act on ψ only + not A. Fix via IBP.

$$V_{int} = [\tilde{KE}] - [KE] = \int dr \frac{e}{2mi}\left[(\nabla\psi^\dagger)\cdot A\psi - \psi^\dagger A\cdot(\nabla\psi)\right] + \frac{e^2A^2}{2m}\psi^\dagger\psi$$

$$= \int dr\, A\cdot\frac{e}{2mi}\left[\psi^\dagger(\overleftarrow{\nabla} - \overrightarrow{\nabla})\psi\right] + \frac{e^2A^2}{2m}\psi^\dagger\psi$$

$$= \int dr\left[-A\cdot\psi^\dagger \vec{j}\,\psi + \frac{e^2A^2}{2m}\psi^\dagger\psi\right]$$

(1) lowest order coupling comes from the first term, so we concentrate on it.

(2) How to get matrix elements?

Use the fact that we can decompose ③

$\psi^{+}(r)$ and $\psi(r)$ into the single particle orbitals. For the case of a semiconductor we have:

conduction band   valence band

$$\psi^{+}(r) = \sum_{k} \phi_{c,k}(r)\, d_k^{+} + \sum_{k} \phi_{v,k}(r)\, c_k^{+}$$

these are e.g. Linea combinations of atomic orbitals

$$\sum_{r_i} e^{ik\cdot r_i}\, \Phi_c^{*}(r-r_i)$$

$\psi^{+} j \psi \Rightarrow$ therefore has combinations connecting atomic orbital in same band and in different bands. Since we are intersted in the case where conduction band is fully empty and valence band fully occupied, only possible excitations will be of the band mixing type $\Rightarrow$ ME like

$$\sum_{\substack{k_1 r_i \\ k_2 r_j}} e^{ik_1\cdot r_i}\, \Phi_c^{*}(r-r_i)\, d_{k_1}^{+}\; j(r)\; e^{-ik_2\cdot r_j}\, \Phi_v(r-r_j)\, c_{k_2}$$

$$= \sum_{\substack{k_1 r_i \\ k_2 r_j}} \left(e^{ik_1\cdot r_i} d_{k_1}^{+}\right) \underbrace{\left\langle \Phi_c^{*}(r-r_i)\, j(r)\, \Phi_v(r-r_j)\right\rangle}_{\text{inter-orbital transition Matrix element}} \left(e^{-ik_2 r_j} c_{k_2}\right)$$

(4)

Assuming that $A(r)$ varies slowly on atomic scale $\Rightarrow$ OK for all but x-rays

We can perform the $\int dr$ to obtain

$$\int dr \; \underbrace{\langle \Phi_c^*(r_j - r_i)\, j(r)\, \Phi_v(r - r_1) \rangle}_{\text{this function is strongly peaked at } r_i = r_j \text{ when } r \sim r_i} A(r)$$

$$\underbrace{\int dr \; \langle \Phi_c^*(r - r_i) j(r) \Phi_v(r - r_i) \rangle}_{J_{c-v}} \; \delta(r_i, r_1) \; A(r_i) \qquad (*)$$

$$\Rightarrow \sum_{\substack{k_1 r_i \\ k_2 r_j}} J_{c-v} \left(e^{ik_1 \cdot r_i} d^\dagger_{k_1} \right) \left(e^{-ik_2 \cdot \cancel{r_1}\, r_i} c_{k_2} \right) \cancel{\delta(r_i, r_j)}\, A(r_i)$$

$$= \sum_{r_i} d^\dagger_{r_i} c_{r_i} J_{c-v} A(r_i)$$

$$= \sum_{k_1 k_2} d^\dagger_{k_1} c_{k_2} A(k_1 - k_2) J_{c-v}$$

$$V_{int} = \sum_{k_1 k_2} \left[d^\dagger_{k_1} c_{k_2} A(k_1 - k_2) J_{c-v} + h.c. \right]$$

Note: $(*)$ can connect $r_i + r_{i\pm1}$, $r_i + r_{i\pm2}$, ...

~~These types of connection will result in ...~~

Just like hopping matrix elements in the tight binding model. Result of this is momentum dependence of J_{c-v} matrix element just like momentum dependence of the dispersion in the tight binding model.

To summarize

$F_{\mu\nu}F^{\mu\nu}$ part

$$H = H_0 + V_{int} = \sum_k \omega_k c_k^\dagger c_k + \sum_k \tilde{\omega}_k d_k^\dagger d_k + \sum_k c|k| a_k^\dagger a_k$$

$$+ \sum_{k_1 k_2} \left[J_{c-v} \, d_{k_1}^\dagger c_{k_2} A(k_1 - k_2) + h.c. \right]$$

$$A(k_1 - k_2) = (NF)\left(a_{k_1 - k_2} - a^\dagger_{k_2 - k_1}\right)$$

In further calculations, we combine J_{c-v} and (NF) into the single factor M

$$V_{int} = \sum_{p,q} M \left[c_p^\dagger d_{p+q} (a_{-q} - a_q^\dagger) + d_{p+q}^\dagger c_p (a_q - a_{-q}^\dagger) \right]$$

$$= \sum_{p,q} M \left(c_p^\dagger d_{p+q} - d_{p-q}^\dagger c_p \right)\left(a_q - a_q^\dagger \right)$$

Let us now compute the photon's Green function

$$G_K^{photon}(t,t') = -i\langle T\, a_K(t)\, a_K^\dagger(t')\rangle$$
$$= \sum_{n=0}^{\infty} \frac{(-i)^{n+1}}{n!} \int_{-\infty}^{\infty} dt_1 \dots dt_n \frac{{}_0\langle T\, \hat{a}_K(t)\, \hat{V}(t_1)\, \hat{V}(t_2) \dots \hat{V}(t_n)\, \hat{a}_K^\dagger(t')\rangle_0}{{}_0\langle S(\infty,-\infty)\rangle_0}$$

first, let us concentrate on the numerator:

$$n=0: \quad -i\langle T\, \hat{a}_K(t)\, \hat{a}_K^\dagger(t')\rangle \equiv -i e^{-ic|K|(t-t')}\, \theta(t-t')$$

$$n=1: \quad (-i)^2 \int_{-\infty}^{\infty} dt_1\, {}_0\langle T\, \hat{a}_K(t) \int dp\, dq\, M_{p,q} \left(\hat{c}_p^\dagger(t_1)\hat{d}_{p+q}(t_1) - \hat{d}_{p-q}^\dagger(t_1)\hat{c}_p(t_1)\right) \times$$
$$\left(\hat{a}_{-q}(t_1) - \hat{a}_q^\dagger(t_1)\right)\hat{a}_K^\dagger(t')\rangle_0$$

Apply Wick's theorem ⇒ separate particle types

$$\Rightarrow -\int_{-\infty}^{\infty} dt_1 \int dp\, dq\, {}_0\langle T\, \underline{\hat{a}_K(t)\left[\hat{a}_{-q}(t_1) - \hat{a}_q^\dagger(t_1)\right]}\, \hat{a}_K^\dagger(t')\rangle \left[\langle \hat{c}_p^\dagger(t_1)\hat{d}_{p+q}(t_1) - \hat{d}_{p-q}^\dagger(t_1)\hat{c}_p(t_1)\right.$$

these are also zero since we change # bosons between initial + final state with odd # of $a, a^\dagger$ operators.

those are zero since # C-fermions in initial + final state diff.

⇒ No linear in A term.

$$n=2: \quad (-i)^3 \int_{-\infty}^{\infty} dt_1\, dt_2 \int dp_1\, dq_1\, dp_2\, dq_2\, {}_0\langle T\, \hat{a}_K(t)\left[\hat{a}_{-q_1}(t_1) - \hat{a}_{q_1}^\dagger(t_1)\right]\left[\hat{a}_{-q_2}(t_2) - \hat{a}_{q_2}^\dagger(t_2)\right]\hat{a}_K^\dagger(t')\rangle_0$$
$${}_0\langle T\left(\hat{c}_{p_1}^\dagger(t_1)\hat{d}_{p_1+q_1}(t_1) - \hat{d}_{p_1-q_1}^\dagger(t_1)\hat{c}_{p_1}(t_1)\right)\left(\hat{c}_{p_2}^\dagger(t_2)\hat{d}_{p_2+q_2}(t_2) - \hat{d}_{p_2-q_2}^\dagger(t_2)\hat{c}_{p_2}(t_2)\right)\rangle_0$$